Heterogeneity Induced Order in Globally Coupled Chaotic Systems

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Abstract

Collective behavior is studied in globally coupled maps with distributed nonlinearity. It is shown that the heterogeneity enhances regularity in the collective dynamics. Low-dimensional quasiperiodic motion is often found for the mean-field, even if each element shows chaotic dynamics. The mechanism of this order is due to the formation of an internal bifurcation structure, and the self-consistent dynamics between the structures and the mean-field.

05.45+b,05.90+m,87.10+e

Dynamics of globally coupled systems has been extensively and intensively studied [1-10]. Such problems naturally appear in physical and biological systems. Coupled Josephson junction array [2] and nonlinear optics with multi-mode excitation [3,4] give such examples, while relevance to neural and cellular networks has been discussed [5]. Among others, study of globally coupled chaotic systems has revealed novel concepts such as clustering, chaotic itinerancy, partial order, and hidden coherence.

In particular, study of collective dynamics has gathered much attention [8-13]. An ensemble of chaotic elements is found not to obey the law of large numbers and to form some kind of coherence [7]. Sometimes, quasiperiodic collective dynamics has been found even if each element shows chaotic dynamics [7,9,10].

In these recent studies, elements are homogeneous, in other words, identical elements are coupled. However, in many systems elements are heterogeneous. In Josephson junction array, each unit is not identical. In an optical system, the gain of each mode depends on its wavenumber. In a biological system, each unit such as a neuron, or a cell is heterogeneous. So far the study of a coupled system with distributed parameters is restricted to synchronization of a non-chaotic system [11]. Thus it is important to check how the notions constructed in globally coupled dynamical systems can be applicable to a heterogeneous case. In the present Letter we demonstrate that the collective order emerges in a heterogeneous system through self-consistent dynamics between the mean-field and an internal bifurcation structure. Here we adopt a globally coupled map with a distributed parameter;

$$x_{n+1}(i) = (1-\epsilon)f_i(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f_j(x_n(j)), \tag{1}$$

$$i = 1, 2, 3, \cdots, N$$

where $x_n(i)$ is the variable of the ith element at discrete time n, and $f_i(x(i))$ are the internal dynamics of each element. For the functions we choose the logistic map

$$f_i(x) = 1 - a(i)x^2,$$

where the parameter $a(i)$ for the nonlinearity is distributed between $[a_0 - \frac{\Delta a}{2}, a_0 + \frac{\Delta a}{2}]$ as $a(i) = a_0 + \frac{\Delta a(2i-N)}{2N}$. We note that the essentially same behavior is found when $a(i)$ is randomly distributed in an interval or the coupling $\epsilon(i)$ is distributed instead of a.

When elements are identical with $\Delta a = 0$, the present model reduces to a globally coupled map (GCM) studied extensively. In this case there appears hidden coherence even when $x_n(i)$ shows chaotic oscillation and elements are totally desynchronized. Indeed the variance of the mean field

$$h_n = \frac{1}{N}\sum_{j=1}^{N} f_j(x_n(j))$$

remains finite as the system size is increased, in contrast with the expectation of the law of large numbers. The mean square deviation (MSD) of the mean-field fluctuation, given by $\langle(\delta h)^2\rangle = \langle(h - \langle h\rangle)^2\rangle$, decreases proportional to N^{-1} up to a crossover size N_c, but then remains constant with the further increase of N.

When elements are not identical, one might expect that hidden coherence would be lost and the law of large number could be recovered. On the contrary, anomalous recovery of the MSD is found with the increase of the system size [8]. This suggests that some kind of order emerges in a heterogeneous system. In the present Letter we clarify the origin of such collective order.

First we begin with the behavior of the mean field fluctuation in our system. In Fig.1, MSD is plotted with the increase of the system size N. Roughly speaking the MSD measures the amplitude of the mean field motion. As the system size increases, the MSD decreases up to a certain size and then stays constant or increases to a certain constant. This result implies the existence of some structure and coherence in the mean field dynamics. The question we address here is their form and origin.

Fig.2(a) gives an example of the return map of the mean field. Here the width of scattered points along the one-dimensional curve decreases with N. Hence the figure clearly shows that the mean field dynamics is on a 2-dimensional torus. The power spectrum of the mean field time series also supports that the motion is quasi-periodic. In this case, the oscillation

has a frequency about 0.435.

From this, one can conclude that there appears collective order in our system with the increase of the size. With the change of a_0, Δa, or ϵ, the mean-field dynamics shows the bifurcation from torus to chaos accompanied by phase lockings. Further bifurcation proceeds to much higher-dimensional chaos (while some structure is still kept). We have also observed the doubling of torus(Fig.2(b)) and other routes to chaos from quasiperiodic motions [13].

There are two cases for the collective motion, although for both cases each element oscillates chaotically without mutual synchronization. In one case (given in Fig.2(a)), all Lyapunov exponents are positive even if collective motion is quasiperiodic. In the other case (given in Fig.2(b)(c))some exponents are negative, although most of them are positive. In this case, the origin of collective order is much easier to be detected.

It should be noted that such low-dimensional collective dynamics is hardly observed in a globally coupled logistic map of identical parameters. With a global scan of the parameter space, such low-dimensional collective dynamics is not observed except for the trivial case, i.e., "direct product" of periodic motions at a window, possible only for a narrow parameter region with a very small coupling (e.g., $a \approx 1.8, \epsilon \approx .01$ it is 3-clustered motion). Thus the heterogeneity in the parameter is essential to form the low-dimensional collective dynamics.

Hereafter we show how this heterogeneity-induced order is possible (mainly for the case with some negative Lyapunov exponents). An example of the return map is given in Fig.2(b). The scenario to be presented consists of two parts. First, we demonstrate the formation of internal bifurcation structure, made possible by the distribution of parameters, which leads to the self-consistent relation between each dynamics and the mean-field. Second, it is shown that the self-consistent dynamics is formed between the motion of the internal bifurcation structure and the mean-field dynamics.

First we study the formation of the internal bifurcation structure. In our system nonlinear parameters are distributed over elements. Dynamics of the ith element depends on the parameter $a(i)$. Hence it is relevant to draw the motion versus the parameter a. Fig.3 gives snapshot patterns of $x_n(a)$ for the period-3 locking in the mean-field. It looks like

an ordinary bifurcation diagram plotted against the change of external parameters, but the patterns of Fig.3 are just snapshot representations of one system consisting of N elements, which is why we call the structure as internal bifurcation.

With the increase of $a(i)$, tangent bifurcation, period doubling, and crisis are observed in the snapshot pattern, if it is viewed as the transition of attractor with the change of control parameters. If the mean-field were an external parameter for each element independent each other, such viewpoint would hold. The mean-field, in our case, is organized self-consistently from each element dynamics, where the heterogeneity is the origin of the internal bifurcation. The organization of the low-dimensional bifurcation structures from a high-dimensional system is a key concept for the collective dynamics as will be discussed.

In Fig.3 elements are almost synchronized for $1.85 < a(i) < 1.887$. The motion of these elements is almost period-3. Indeed, this motion comes from the window in the single logistic map, which plays an important role to understand the mechanism of collective behavior. Since the period-3 window is most prominent in the logistic map, we explain the mechanism for this case, although the explanation can be applied generally to other windows.

In the logistic map $f(x) = 1 - ax^2$, the period-3 window appears through a tangent bifurcation of the 3rd iterate of the map $y = f(f(f(x)))$. In this case, the map is tangential to $y = x$ at 3 points corresponding to the periodic points. On the other hand, for the mapping $f_{\delta_n}(x) = 1 - ax^2 + \delta_n$ with a time-dependent parameter δ_n, the bifurcation usually occurs when the 3rd iterate $y = f_{\delta 3}(f_{\delta 2}(f_{\delta 1}(x)))$ is tangential to $y = x$ only at one point, unless there is certain restriction to the external field δ. In other words one specific phase of the period-3 oscillation is selected in accordance with the external parameter.

In Fig.3, the tangent bifurcation happens only at one point around $a \approx 1.85$. As the parameter a is larger, the period doubling bifurcation and consequently crisis occurs beyond which elements fall into a fully chaotic state. Thus synchronized and fully desynchronized motions coexist.

This internal structure forms the mean-field as the "input" to each element self-consistently. Period-3 clustered motion and fully desynchronized motion form the mean-field

locked to period-3. On the other hand, this period-3 "input" to each element forms the internal structure mentioned above. This self-consistent formation of clustered motion leads to the simplest form of the collective order. We note that such self-consistency is not attained in a homogeneous system, except for a trivial case where all elements are synchronized as a motion of a periodic-window.

When the coupling strength ϵ is smaller, another tangent bifurcation occurs. This bifurcation occurs after the complete formation of the period-3 clustered motion mentioned above. As the 2nd clustered motion is formed, the mean-field is varied, which changes the internal bifurcation structure. Then the period-3 locking in the mean-field collapses, when we need the second scenario for the self-consistent dynamics between the mean-field and the internal bifurcation structure.

The scenario is summarized as follows: A coherent cluster is formed for some parameter values of $a(i)$, by using a window structure in the internal bifurcation, as is discussed. Then the number of elements belonging to this cluster increases, which leads to a long-term change of the mean-field. This, then, destabilizes the cluster, but stabilizes another cluster with a different phase of oscillation. The latter cluster again gathers elements, which then changes the mean-field in the opposite direction. Repeating this process, a long-term quasiperiodic oscillation is self-consistently formed.

As the simplest example, we discuss the quasi-periodic case given by Fig.2(c). Fig.4 shows the snapshots of $x(a)$ corresponding to the case of Fig.2(c).

To see the above scenario, we study the change of internal structure of $x(a)$ at $3n$ step. The process through Fig.4 is summarized as follows. Two-clustered motion is formed, although it does not last stably. The 1st cluster at $x = 1$ (denoted as $c+$) breaks down near $a = 1.917$ by crisis, and the elements whose $a(i)$'s are larger than this value leave the cluster. On the other hand, after the formation of the 1st cluster, the tangent bifurcation near $x = -0.7$ and $a = 1.92$ occurs. It forms the 2nd cluster (denoted as $c-$), which attracts elements (Fig. 4(a)), while the 1st cluster at $x = 1$ starts to collapse from smaller values of a successively. With this process, the 2nd cluster grows from lager a to smaller (Fig.4(b)(c)).

With the complete collapse of the 1st cluster, the 3rd cluster is formed at $x = 0$ (denoted as $c0$) (Fig.4(d)).

The above process, taking about 170 time steps, repeats successively changing the roles of the three clusters $c+$, $c0$, and $c-$. This repeated collapse and formation of the three clusters is the origin of the quasi-periodic motion in the mean field.

To see our feedback scenario here, we need to clarify (a) how the internal structure determines the mean-field and (b) how the mean-field modifies the stability of clusters. This is carried out by analyzing the change of

$$x_{3n} = F_{h_{3n-1}}(F_{h_{3n-2}}(F_{h_{3n-3}}(x))), \tag{2}$$

which is the 3rd iterate of $F_{h_n}(x) = (1-\epsilon)(1-ax^2) + \epsilon h_n$, where h_n is the mean field as an external parameter for each element.

Let us consider the relationship between $c0$, $c+$, $c-$-clusters and h_{3n}, h_{3n-1}, h_{3n-2}. The step (a) is rather simple. When, for example, the cluster $c+$ grows, $\frac{1}{N}\sum_{i=1}^{N} x_{3n}(i)$ is increased. Then h_{3n-1} is increased, since

$$h_{n-1} = \frac{1}{N}\sum_{i=1}^{N} f_i(x_{n-1}(i)) = \frac{1}{N}\sum_{i=1}^{N} x_n(i).$$

By a cyclic rotation of $c0$, $c+$, $c-$, with the mapping from 3n to 3n+1 etc, other relationships between h_{3n} and the cluster structure are obtained.

On the other hand, a straightforward calculation of eq.(2) tells us that the (in)stability of c_+, c_0 or c_- cluster mainly depends on h_{3n-1}, h_{3n-3} or h_{3n-2} respectively. Each cluster is stabilized as h_{3n-1}, h_{3n-3} or h_{3n-2} gets larger respectively. Thus the process (b) is obtained. From (a) and (b), the stability of each cluster is mainly governed by the change of the distribution of x_{3n}, x_{3n-2} and x_{3n-1} respectively.

Let us reconsider the above scenario in more detail. After the formation of the c_+-cluster at $3n$ step, the c_--cluster starts to be stabilized from lager a. Then the elements that left the c_+ by the crisis at $a = 1.917$ are absorbed by c_-. This formation of the c_- cluster makes the mean-filed h_{3n-1} to decrease(Fig.4(a)).

On the other hand, this decrease of the mean-field modifies the stability of the c_+-cluster; the tangent bifurcation point of c_+ moves to larger a. Then the c_+ cluster is destabilized from smaller a till it collapses(Fig.4(b)(c)).

Corresponding to the collapse of c_+-cluster at $3n$ step, c_- cluster starts to collapse at $3n-2$ step. According to this, h_{3n-3} starts to increase, which stabilizes the c_0 cluster at $3n$ step. Now h_{3n-1} starts to increase(Fig.4(d)), and so forth.

We note that the above feedback process between the mean-field and internal bifurcation structure is possible, since the value of a is non-identical. The role of elements is differentiated as to the synchronization and desynchronization, which temporally changes as in the case for chaotic itinerancy [1,14,15]. We also note that a slow modulation of the mean-field dynamics is formed by the feedback. This separation of time scales is necessary to have a low-dimensional collective order; otherwise high-dimensional chaotic dynamics remains in the mean-field as in the hidden coherence in GCM [7].

Although we have explained the above scenario for the period-3 window case due to its simplicity, this mechanism is generally applied to our system, since each (logistic) dynamics contains a variety of windows and bifurcations. For example, we have seen the change of the synchronization and internal bifurcation structures for the parameters for Fig.2(a), where all Lyapunov exponents are positive and clear windows are not visible.

To sum up, we have shown the formation of low-dimensional collective dynamics in a coupled chaotic system with heterogeneity. The mechanism of the formation is due to the internal bifurcation structure afforded by heterogeneity, and the self-consistent feedback dynamics between the mean-field and synchronization of elements. We note that this mechanism is expected to be quite general, as long as each local dynamics allows for bifurcations with the change of some parameters, distributed by elements. Thus our scenario for the collective order can be observed in coupled systems such as Josephson junction arrays, and multi-mode laser systems, as well as in biological networks.

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FIGURES

FIG. 1. Mean square deviation (MSD) of the distribution of the mean-field h is plotted with the increase of the system size $N.a = 1.9, and\Delta a = 0.05$, while the parameter ϵ is shown at the right.

FIG. 2. Return map of the mean-field h. $a_0 = 1.9$ (a)$\Delta a = 0.05$, $\epsilon = 0.11$, $N = 8 \times 10^6$. (b)$\Delta a = 0.1$, $\epsilon = 0.053$, $N = 2^{18}$. (c)$\Delta a = 0.1$, $\epsilon = 0.05$, $N = 2^{21}$.

FIG. 3. Internal Bifurcation Diagram. $x_n(i)$ is plotted versus $a(i)$. Here the mean-field is locked to period-3. $a_0 = 1.9$, $\Delta a = 0.1$, $\epsilon = 0.055$. Time step 5000 (a), and 5001 (b). At the next iterate, the coherent structure of $x_n(a)$ for $a < 1.88$ moves to $x \approx 1$, while another iterate leads to the structure of Fig.3(a)

FIG. 4. Dynamics of the internal bifurcation structure. $a_0 = 1.9$, $\Delta a = 0.1$, $\epsilon = 0.053$, at time step 5000 (a), 5081 (b), 5126 (c), 5271(d).

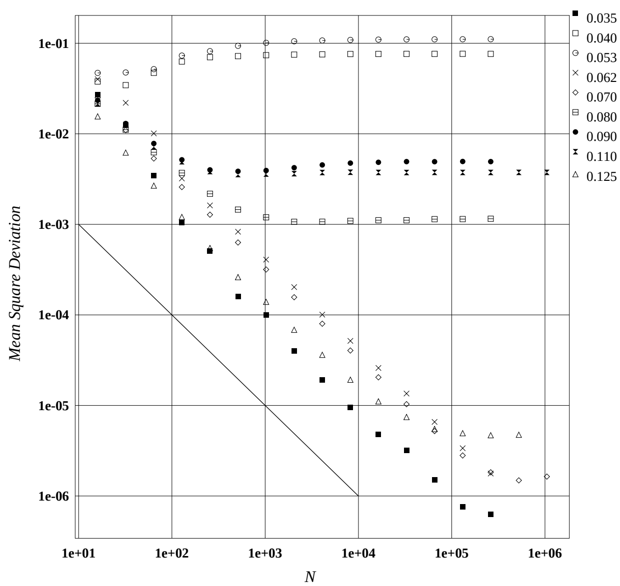

Fig. 1 T. Shibata & K. Kaneko

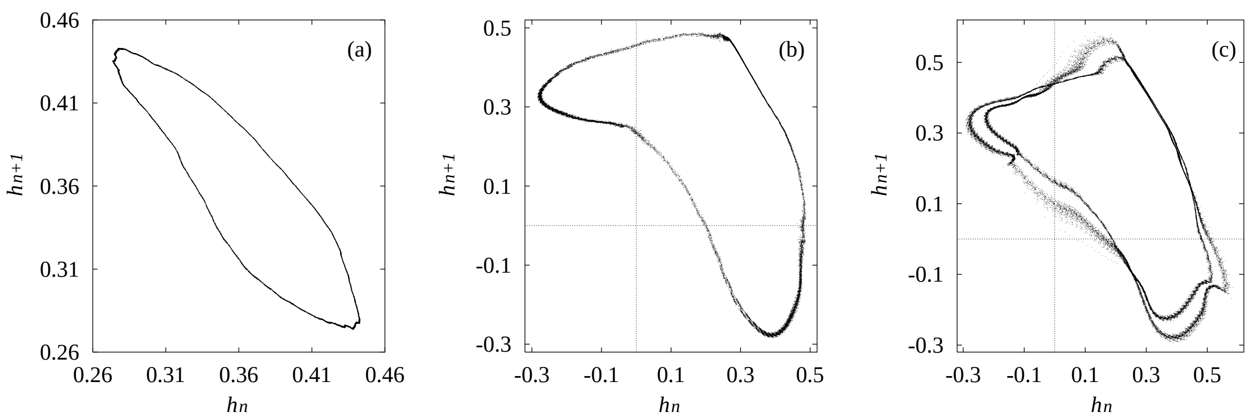

Fig.2 T. Shibata & K. Kaneko

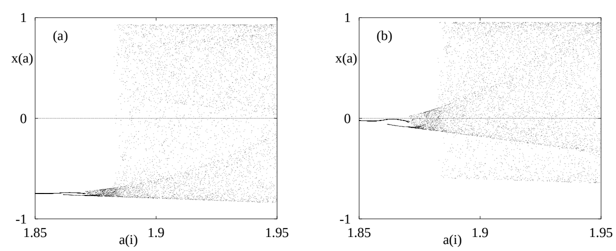

Fig.3 T. Shibata & K. Kaneko

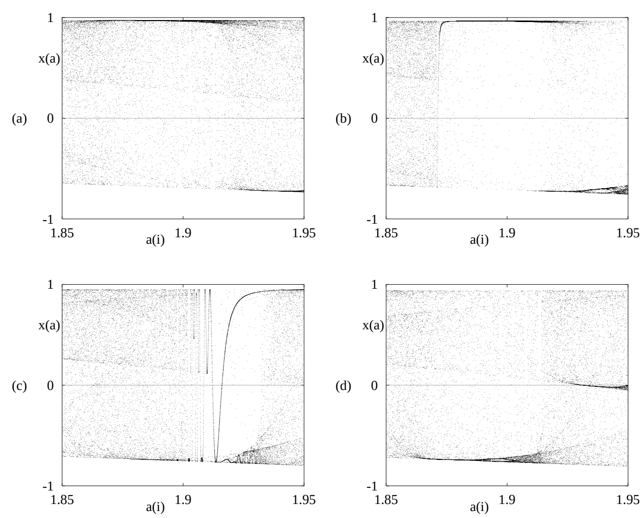

Fig.4 T. Shibata & K. Kaneko